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# CONES OF ALTERNATING AND CUT SUBMODULAR SET FUNCTIONS

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We describe facets of the cones of alternating set functions and cut submodular set functions generated by directed and undirected graphs and by uniform even hypergraphs. This answers a question asked by L. Lovász at the Bonn Mathematical Programming Conference in 1982. We show that there is a network flow algorithm for minimizing a hypergraph cut set function.

## 1. Introduction

On the Bonn Mathematical Programming Conference 1982 L. Lovász [4] asks for the valid inequalities for the cone of cut submodular set functions. If $G$ is a directed graph with arcs $[i,j]$ having nonnegative weights $c[i,j]$, the cut set function $f$ generated by $G$ has the value $f(S)=\sum_{i\in S,\, j\notin S} c[i,j]$ on a subset $S$ of the vertex set $V$ of $G$. If the weights $c[i,j]$ are symmetric, $c[i,j]=c[j,i]$ for all arcs $[i,j]$, then $f$ is a symmetric cut function of an undirected graph.

The polyhedral cone $\mathscr{R}_0(V)$ of cut functions lies in the finite dimensional space $\mathbf{R}^{B(V)}$ of all set functions defined on the family $\mathscr{B}(V)=2^V$ of all subsets of a set $V$. Extreme rays of $\mathscr{R}_0(V)$ are functions $w_{[i,j]}$ generated by graphs with only one arc $[i,j]$ of unit weight.

We show that $\mathscr{R}_0(V)$ is the intersection of the cone $\mathscr{C}_+(V)$ of all nonnegative submodular functions with the space $D_0(V)\subseteqq\mathbf{R}^{\mathscr{B}(V)}$ spanned by all vectors $w_{[i,j]}$. The cone $\mathscr{R}_2(V)\subseteqq\mathscr{R}_0(V)$ of symmetric cut functions is the intersection of the cone $\mathscr{C}(V)$ of all submodular functions with the space $D_2(V)\subseteqq D_0(V)$ spanned by vectors $w_{i,j}=w_{[i,j]}+w_{[j,i]}$.

Where $\mathscr{R}_m(V)$ is a cone of cut functions generated by cuts of a complete uniform $m$-hypergraph, and extreme ray $w_A$ of the cone is generated by a hypergraph with only one edge $A$ of the cardinality $|A|=m$. The dimension of the space $D_{2k}(V)$ spanned by $\mathscr{R}_m(V)$ with $m=2k$ is equal to the number of all extreme rays of the cone $\mathscr{R}_{2k}(V)$.

The space $D_{2k}(V)$ for $k\geqq 0$ lies in the intersection of hyperplanes defined by alternating equalities. The facets of the cones $\mathscr{R}_{2k}(V)$ for $k>0$ are defined by alternating inequalities. This is due to the fact that the function $w_A$ for a nonempty $A\subseteqq V$ is a symmetrization of a monotone submodular (0, 1)-function $q_A$ which is an extreme ray of the cone $\mathscr{A}_0(V)$ of alternating set functions. The cone $\mathscr{A}_0(V)$ was studied by G. Choquet [1].

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Here we give all facets of the cones $\mathscr{A}_0(V)$ and $\mathscr{R}_{2k}(V)$ for $k \geqq 0$ and show that the cones $\mathscr{R}_{2k}(V)$for a fixed $k \geqq 0$ have a co-NP description.

In the last section we consider some functions of cuts separating two specified vertices of a hypergraph. We show that there is a network flow algorithm for minimizing such a function.

In what follows we denote $V-X$ by $\bar{X}$ and the cardinality of $X$ by $|X|$.

## 2. Alternating functions

G. Choquet [1] introduced an alternating operator $D_f^k$ on a set function $f \in \mathbf{R}^{\mathscr{B}(V)}$ (also, see [5])

$$D_f^k(X, A_i\colon i \in S) = \sum_{T \subseteq S} (-1)^{|T|} f(X \cup A(T)),$$

where $X$ and $A_i$ $(i \in S)$ are subsets of $V$, $A(T) = \bigcup_{i \in T} A_i$, $A(\emptyset) = \emptyset$, and $S$ is an arbitrary finite index set of the cardinality $|S| = k$.

A set function $f$ is called $k$-alternating if $D_f^k(X, A_i\colon i \in S) \leqq 0$ for all $X \subseteq V$ and all $k$-families $\{A_i\colon i \in S\}$, $|S| = k > 0$, i.e. if

$$\sum_{T \subseteq S} (-1)^{|T|} f(X \cup A(T)) \leqq 0. \tag{2.1}$$

A function $f$ is called alternating if $f$ is $k$-alternating for all $k > 0$. The cone $\mathscr{A}_0(V)$ of alternating functions with $f(\emptyset) = 0$ is determined by the inequalities (2.1) for all $X \subseteq V$ and for all families $\{A_i\colon i \in S\}$, $S \neq \emptyset$, and by the equality $f(\emptyset) = 0$.

A function $f$ is 1-alternating iff $f$ is monotone, $f$ is 2-alternating iff $f$ is monotone and submodular. Hence $\mathscr{A}_0(V)$ is a subcone of the $\mathscr{C}_0(V)$ of monotone submodular functions with $f(\emptyset) = 0$.

G. Choquet notes that if $f \in \mathscr{A}_0(V)$ and there are subsets $A, B \subseteq V$ such that $0 \neq f(A) < f(B)$ then $f = f_1 + f_2$ where $f_1(X) = f(X \cup A) - f(A)$ and $f_2(X) = f(X) + f(A) - f(X \cup A)$. It is not difficult to verify that $f_1, f_2 \in \mathscr{A}_0(V)$, $f_1$ and $f_2$ are not in proportion (since $f_1(A) = 0 \neq f(A) = f_2(A)$) and are non-zero functions ($f_1(B) = f(B \cup A) - f(A) \geqq f(B) - f(A) > 0$). Therefore every function lying on an extreme ray of $\mathscr{A}_0(V)$ can have only one non-zero value, i.e. these functions are (0, 1)-functions up to a multiple.

Every (0, 1)-function $f \in \mathscr{C}_0(V)$ is determined by its upper zero $B = \bigcup \{X \subseteq V\colon f(X) = 0\}$. Let $q_C$ be a such a function with upper zero $B = V - C = \bar{C}$, i.e.

$$q_C(X) = \begin{cases} 1 & \text{if} \quad X \cap C \neq \emptyset \\ 0 & \text{if} \quad X \subseteq \bar{C}. \end{cases} \tag{2.2}$$

Let $\delta(A, B) = 0$, if $A \neq B$, and $\delta(A, B) = 1$ if $A = B$.

**Lemma 2.1.** *The functions $q_C$ belong to the cone $\mathscr{A}_0(V)$, and*

(2.3)
$$D_{q_C}^k(X, A_i\colon i \in S) = \sum_{T \subseteq S} (-1)^{|T|} q_C(X \cup A(T)) = \begin{cases} -\delta(T_C, S) < 0 & \textit{if} \quad X \subseteq \bar{C} \\ 0 & \textit{otherwise} \end{cases}$$

*where* $T_C = S - \bigcup \{T \subseteq S\colon X \cup A(T) \subseteq \bar{C}\}$.

**Proof.** By (2.2), it is easy to see that $q_C(X\cup A(T))=1$ if $X\nsubseteq \bar{C}$ and $q_C(X\cup A(T))= =q_{T_C}(T)$, where $q_{T_C}\in\mathscr{C}_0(S)$, if $X\subseteq\bar{C}$. Hence, for $S\neq\emptyset$ we have

$$\sum_{T\subseteq S}(-1)^{|T|}q_C(X\cup A(T)) = \sum_{T\subseteq S}(-1)^{|T|} = 0$$

if $X\nsubseteq\bar{C}$, and $= \sum_{T\subseteq S}(-1)^{|T|}q_{T_C}(T)= \sum_{T\subseteq S}(-1)^{|T|}-\sum_{T\subseteq T_C}(-1)^{|T|}=-\delta(\bar{T}_C,\emptyset)= = -\delta(T_C,S)$. ∎

We use (2.3) for $\emptyset\neq S\subseteq V$, $X=\bar{S}$ and $A_i=\{i\}$. In the case, $X\subseteq\bar{C}$ means $\bar{S}\subseteq\bar{C}$, and $T_C=S$ means $S=C$. Hence we have

$$\sum_{T\subseteq S}(-1)^{|T|}q_C(\bar{S}\cup T) = -\delta(C,S). \tag{2.4}$$

It follows that the $2^n-1$ functions $q_A$ for $A\subseteq V$, $A\neq\emptyset$, constitute a basis of the space $\mathbf{R}_0^{\mathscr{B}(V)}$ of all set functions $f\colon \mathscr{B}(V)\to\mathbf{R}$ with $f(\emptyset)=0$. Let $f=\sum_A g_f(A)q_A$ be the representation of $f\in\mathbf{R}_0^{\mathscr{B}(V)}$ in the basis $\{q_A\}$. Using (2.4) we can find coefficients $g_f$:

$$g_f(A) = \sum_{B\subseteq A}(-1)^{|B|+1}f(\bar{A}\cup B) = \sum_{\bar{A}\subseteq B\subseteq V}(-1)^{|B-\bar{A}|+1}f(B). \tag{2.5}$$

**Theorem 2.2.** *The cone $\mathscr{A}_0(V)$ of alternating functions has $2^n-1$, $n=|V|$, extreme rays, which are all $2^n-1$ functions $q_A$ for $\emptyset\neq A\subseteq V$. The facets of the cone $\mathscr{A}_0(V)$ are defined by the $2^n-1$ inequalities*

$$\sum_{A\subseteq B\subseteq V}(-1)^{|B-A|}f(B)\leqq 0,\quad V\neq A\subseteq V, \tag{2.6}$$

*which are equivalent to all the inequalities* (2.1) *for $f$ with $f(\emptyset)=0$.*

**Proof.** It was shown in [3] that every function $q_A$ is an extreme ray of the cone $\mathscr{C}_0(V)\supseteq\mathscr{A}_0(V)$. Recall that every (0, 1)-function $f\in\mathscr{C}_0(V)$ coincides with one of $q_A$'s. Therefore, according to the above Choquet remark and Lemma 2.1, $q_A$ are extreme rays of $\mathscr{A}_0(V)$ and there are no other extreme rays. Since $f\in\mathscr{A}_0(V)$ iff $g_f(A)\geqq 0$ for all $A$, by (2.5) we obtain that (2.6) are facets of $\mathscr{A}_0(V)$. ∎

## 3. Cut functions of a directed graph

Recall that a set function $f\in\mathbf{R}^{\mathscr{B}(V)}$ is called a cut function if there are nonnegative weights $c[i,j]$ of arcs $[i,j]$ of a complete directed graph $K_n^{\mathrm{or}}$ with a vertex set $V$, $|V|=n$, such that for $X\subseteq V$ and $\bar{X}=V-X$

$$f(X) = \sum_{i\in X,\, j\in\bar{X}} c[i,j]. \tag{3.1}$$

The cut functions are submodular and compose a polyhedral cone $\mathscr{R}_0(V)\subseteq \subseteq\mathbf{R}^{\mathscr{B}(V)}$. If the weights $c[i,j]$ are symmetric, i.e. if $c[i,j]=c[j,i]$ then so is $f$, that is $f(X)=f(\bar{X})$ for all $X\subseteq V$. The symmetric functions constitute a subcone $\mathscr{R}_2(V)\subseteq\mathscr{R}_0(V)$ of cut functions of an undirected complete graph $K_n$. The cut functions with $c[i,j]\neq 0$ only for $[i,j]\in E\subseteq E_n$ ($E_n$ is the set of all arcs of $K_n^{\mathrm{or}}$) constitute a cone $\mathscr{R}_0(V,E)\subseteq\mathscr{R}_0(V)\equiv\mathscr{R}_0(V,E_n)$ of functions generated by a graph

$G(V,E)$. The cone $\mathcal{R}_2(V,E)\subseteq\mathcal{R}_2(V)$ of symmetric functions is defined analogously, with $E\subseteq V^2$, where $V^2$ is the set of all unordered pairs $(i,j)$ of elements of $V$, $|V^2|=\binom{n}{2}$.

Let $w_{[i,j]}$ be a cut function related to a graph with only one arc $[i,j]$ of unite weight $c[i,j]=1$, such that

$$w_{[i,j]}(X)=\begin{cases}1 & \text{if } i\in X,\ j\in \bar{X}\\ 0 & \text{otherwise.}\end{cases}$$

Since every cut function can be represented as

$$f(X)=\sum_{[i,j]\in E} c[i,j]w_{[i,j]}(X),\quad c[i,j]\geqq 0,\quad X\subseteq V, \tag{3.2}$$

$w_{[i,j]}$ are extreme rays of the cone $\mathcal{R}_0(V,E)$.

It is not difficult to verify that

$$w_{[i,j]}=q_{i,j}-q_i,$$

where $q_{i,j}$ and $q_i$ are $q_A$ of the preceding section for $A=\{i,j\}$ and $\{i\}$.

Extreme rays of the cone of symmetric cut functions are the functions

$$w_{i,j}=w_{[i,j]}+w_{[j,i]}=2q_{i,j}-q_i-q_j.$$

Obviously the cones $\mathcal{R}_0(V)$ and $\mathcal{R}_2(V)$ lie in a subspace $D_0(V)\subseteq\mathbf{R}_0^{\mathcal{B}(V)}$ spanned by vectors $w_{[i,j]}$. The subspace $D_0(V)$ lies in a subspace spanned by vectors $q_i$, $i\in V$, and $q_{i,j}$, $(i,j)\in V^2$. This last subspace is determined by the equations $g_f(A)=0$ for $|A|\geqq 3$, where $g_f(A)$ is given in (2.5). Recall, $g_f(A)$ is the $q_A$-coordinate of a vector $f\in\mathbf{R}_0^{\mathcal{B}(V)}$ in the $q$-basis. Below we give a co-NP description of the space $D_0$.

We apply to $q_{i,j}$ and $q_i$ Lemma 2.1 for $X=\emptyset$ and for a family $\{A_i\colon i\in S\}$ of disjoint sets $A_i$. It is easy to see, that if $|S|\geqq 3$ then for every pair $(i,j)\in V^2$ there is a set $A_k$, $k\in S$, such that $A_k\subseteq V-\{i,j\}$. It follows that cut functions satisfy the equality

$$\sum_{T\subseteq S}(-1)^{|T|}f(A(T))=0,\quad |S|\geqq 3. \tag{3.3}$$

These equalities for $|S|=3$ were found by W. Cunningham [2]. Since for any cut function $f(\emptyset)=0$, (3.3) is valid for $S=\emptyset$.

Consider the equalities (3.3) for one-element sets $A_i$ and define functions

$$a_f(S)=-\sum_{T\subseteq S}(-1)^{|T|}f(T). \tag{3.4}$$

In particular,

$$a_f(i)=f(i),\quad a_f(i,j)=f(i)+f(j)-f(i,j), \tag{3.5}$$

and according to (3.3) $a_f(S)=0$ for $|S|\geqq 3$.

The linear transformation $f\to a_f$ is lower triangular and nondegenerate. Hence the hyperplanes

$$a_f(S)=0,\quad S\subseteq V,\quad |S|\geqq 3\quad\text{and}\quad S=\emptyset \tag{3.6}$$

are linear independent and their intersection is a subspace $D(V)$ of $\mathbf{R}_0^{\mathcal{B}(V)}$ of the dimension $n+n(n-1)/2=n(n+1)/2$. Applying Möbius inversion to (3.4) and using (3.6) we obtain

$$f(S)=-\sum_{T\subseteq S}(-1)^{|T|}a_f(T)=\sum_{i\in S}a_f(i)-\sum_{(i,j)\in S^2}a_f(i,j). \tag{3.7}$$

According to (3.5), this expression in $f$-coordinates has the form

$$f(S)=\sum_{(i,j)\in S^2}f(i,j)-(|S|-2)\sum_{i\in S}f(i),\quad S\subseteq V,\quad |S|\geqq 3,\quad f(\emptyset)=0. \tag{3.8}$$

(3.7) is a representation of a function $f\in D(V)$ in the coordinates $a(i)$, $i\in V$, and $a(i,j)$, $(i,j)\in V^2$, which are the values of the function $a_f$ on one- and two-element sets. The $a$-coordinates of the functions $w_{[i,j]}$ are easy to compute

$$a_{[i,j]}(i)=1,\quad a_{[i,j]}(k)=0\quad \text{if}\quad k\neq i,$$

$$a_{[i,j]}(i,j)=a_{[j,i]}(i,j)=1,\quad a_{[i,j]}(k,m)=0\quad \text{if}\quad (k,m)\neq(i,j).$$

Let $A'$ be a $n(n+1)/2\times|E|$ matrix with elements $a_{[i,j]}(k)$ for $k\in V$ and $-a_{[i,j]}(k,m)$ for $(k,m)\in V^2$, all $[i,j]\in E$. The matrix $A'$ is an incidence matrix of the following bipartite graph $G_w(E)$ with arc set $E$. The vertex set of $G_w(E)$ is a set $V_w=V\cup V^2$ with a partition $V$ and $V^2$. An arc $[i,j]$ connects a vertex $i\in V$ with a vertex $(i,j)\in V^2$ and is directed from $i$ to $(i,j)$. The degree of a vertex $(i,j)\in V^2$ in $G_w(E)$ is less than or equal to 2.

Alternating (3.2) we obtain $a_f(T)=\sum_{[i,j]\in E}c[i,j]a_{[i,j]}(T)$. Taking into account (3.6) we can this write down as

$$a'_f=A'c,\quad c\geqq 0, \tag{3.9}$$

where $a'_f$ is a $n(n+1)/2$-vector of the space $D(V)$ with coordinates $a'_f(i)=a_f(i)$ and $a'_f(i,j)=-a_f(i,j)$. The system (3.9) has the following explicit form

$$a_f(i)=\sum_{j:[i,j]\in E}c[i,j],\quad a_f(i,j)=c[i,j]+c[j,i],\quad c[i,j]\geqq 0,\quad [i,j]\in E.$$

This system regarded as a system of equalities and inequalities with unknown $c[i,j]$ is a supply-demand problem on the bipartite graph $G_w(E)$ with the supplies $a_f(i)$, $i\in V$, the demands $a_f(i,j)$, $(i,j)\in V^2$, and with infinite capacities of the arcs $[i,j]\in E$. The facets of the cone (3.9) define a minimal set of necessary and sufficient conditions that this problem has a solution for a given vector $a_f$. It is well known that these conditions have the form

$$-d(\overline{W},W)\leqq\sum_{i\in W}a_f(i)-\sum_{(i,j)\in W}a_f(i,j)\leqq d(W,\overline{W}), \tag{3.10}$$

where $W\subseteq V_w$, $\overline{W}=V_w-W$, and $d(W,\overline{W})$ is the capacity of the cut $(W,\overline{W})$.

A cut $(W,\overline{W})$ is considered as a set of arcs having one end in $W$ and another end in $\overline{W}$. The capacity $d(W,\overline{W})$ of the cut $(W,\overline{W})$ is the sum of capacities of those arcs of the cut, which are directed from $W$ to $\overline{W}$. A cut is called empty if the set of its arcs is empty. A cut $(W,\overline{W})$ is called oriented (antioriented) if all its arcs are directed from $W$ to $\overline{W}$ (from $\overline{W}$ to $W$, respectively). An empty cut is simultaneously oriented and antioriented. A cut of the graph $G_w$ is called minimal if deleting of its arcs increases the number of connected components of $G_w$ by exactly one.

**Proposition 3.1.** *The cone* (3.9) *lies in a subspace of the space* $D(V)$. *This subspace is defined by the equalities*

$$\sum_{i\in W_0} a_f(i)-\sum_{(i,j)\in W_0} a_f(i,j)=0, \quad W_0\in\mathscr{W}_0 \tag{3.11}$$

*where* $\mathscr{W}_0$ *is the family of all sets of vertices of connected components of* $G_w$.

*The facets of the cone are defined by the following inequalities*

$$\sum_{i\in W} a_f(i)-\sum_{(i,j)\in W} a_f(i,j)\geqq 0, \quad W\in\mathscr{W}_1, \tag{3.12}$$

*where* $\mathscr{W}_1$ *is the family of all sets* $W\subseteqq V_w$, *which determine minimal nonempty oriented cuts* $(W, \overline{W})$ *of* $G_w$.

**Proof.** The capacity of any arc of the net $G_w(E)$ is equal to infinity, so that $d(W, \overline{W})=0$ or $\infty$ for all $W\subseteqq V_w$. The inequalities (3.10) for an empty cut $(W, \overline{W})$ are equivalent to an equality of the type (3.11). Obviously, the family $\mathscr{W}_0$ give us a minimal linearly independent system of such equalities.

The inequalities in (3.10) for a nonempty cut $(W, \overline{W})$ are nontrivial only if the cut is oriented or antioriented. The nontrivial part of (3.10) for an oriented cut has the form (3.12). We can regard only the oriented cuts since (3.10) for an antioriented cut $(W, \overline{W})$ is equivalent to (3.12) for the oriented cut $(\overline{W}, W)$ minus the equality (3.11) for $W_0=V_w$. But again, the family $\mathscr{W}_1$ provide a minimal linearly independent system of such inequalities. ∎

Consider in details two special cases. Let $E=E_n$ at first. The graph $G(V, E_n)=K_n^{\text{or}}$ has two arcs $[i,j]$ and $[j,i]$ for every pair $(i,j)$ of vertices. Note, that $G_w(E_n)$ is connected. Therefore $\mathscr{W}_0=\{V_w\}$ in this case. According to (3.5) the equality (3.11) for $W_0=V_w$ has the form $f(V)=0$. A cut $(W_w, V-W)$ is a minimal oriented cut of $G_w(E_n)$ if either $W=V_w-(i,j)$, or $W=S\cup S^2$ for $S\subseteqq V$, $S^2\subseteqq V^2$, $S\neq\emptyset, V$. By (3.12) these cuts provide facets of the cone $\mathscr{R}_0(V, E_n)$ (up to the equality (3.11) for $W_0=V_w$) of the form

$$a_f(i,j)\geqq 0, \quad (i,j)\in V^2, \quad \sum_{i\in S} a_f(i)-\sum_{(i,j)\in S^2} a_f(i,j)\geqq 0, \quad S\subseteqq V.$$

Thus, using (3.5) we obtain from Proposition 3.1.

**Theorem 3.2.** *The cone* $\mathscr{R}_0(V, E_n)$ *of cut functions over the complete graph* $K_n^{\text{or}}$ *lies in the space* $D_0(V)$, *which is the intersection of the space* $D(V)$ *defined by* (3.8) *with the hyperplane* $f(V)=0$. *The facets of* $\mathscr{R}_0(V, E_n)$ *are determined by the inequalities*

$$f(i)+f(j)-f(i,j)\geqq 0, \quad (i,j)\in V^2$$

$$f(S)\geqq 0, \quad \emptyset\neq S\subsetneqq V. \tag{3.13}$$

*The cone* $\mathscr{R}_0(V, E_n)$ *is the intersection of the cone* $\mathscr{C}_+(V)$ *of all nonnegative submodular functions with the space* $D_0(V)$. ∎

To make sure that $f$ is not in the cone $\mathscr{R}_0(V, E_n)$, it is sufficient to point out a set $S$ for which (3.8) or (3.13) is violated. This fact can be proved with using only polynomially many values of the function $f$. Therefore testing whether $f$ belong to the cone $\mathscr{R}_0(V, E_n)$ is in co-NP. (This fact was noted by referee.)

Now, consider the case $E=E_0$, where exactly one arc from each pair of arcs $[i,j]$ and $[j,i]$ belongs to $E_0$. The graph $G(V,E_0)$ is a tournament. In this case $G_w(E_0)$ has $n=|V|$ components, each of which is an oriented star with a source $i$ for every $i\in V$. By (3.11) these components define the $n$ hyperplanes

$$a_f(i)-\sum_{j:\,[i,j]\in E_0} a_f(i,j)=0,\quad i\in V.$$

The cone $\mathscr{R}_0(V,E_0)$ lies the intersection of these hyperplanes with the space $D(V)$ defined by (3.8). The dimension of the intersection equals $\dim D(V)-n=$ $=n(n-1)/2$. The cone $\mathscr{R}_0(V,E_0)$ has $n(n-1)/2$ extreme rays $w_{[i,j]}$ for $[i,j]\in E_0$, which are linearly independent, since $|E_0|$ is equal to the dimension of the intersection. Thus we have

**Theorem 3.3.** *The cone $\mathscr{R}_0(V,E_0)$ of cut functions over the tournament $G(V,E_0)$ lies in the space defined by the equations* (3.8) *and the equations*

$$(d_i-1)f(i)=\sum_{j:\,[i,j]\in E_0}(f(i,j)-f(j)),\quad i\in V,$$

*($d_i$ is the outdegree of the vertex $i$ in the tournament $G(V,E_0)$).*

*The facets of $\mathscr{R}_0(V,E_0)$ are described by the $n(n-1)/2$ inequalities*

$$f(i)+f(j)-f(i,j)\geqq 0,$$

*which are determined by one-vertex cuts $(V_w-(i,j),(i,j))$ of $G_w(E_0)$.* ∎

The cone $\mathscr{R}_2(V,E)$ of symmetric cut functions is considered below in Section 4.

## 4. Cut functions induced by a hypergraph

Let $H(V,E)$ be a hypergraph with a vertex set $V$ and a hyperedge set $E=\{A\}$, $A\subseteqq V$. Each edge $A\in E$ has a nonnegative weight $c_A$. Every subset $X\subseteqq V$ determines a cut of $H$. Edges of the cut $X$ are those $A\in E$ for which $A\cap X\neq\emptyset$ and $A\cap\bar X\neq\emptyset$. The sum of the weights of the edges of the cut $X$ is a value on $X$ of a symmetric submodular cut function $f$. Thus

$$f(X)=\sum_{A\in E}c_Aw_A(X)$$

where $w_A$ is an elementary cut function, such that

$$w_A(X)=\begin{cases}1 & \text{if}\quad A\cap X\neq\emptyset,\quad A\cap\bar X\neq\emptyset,\\ 0 & \text{otherwise.}\end{cases}$$

Using (2.2), it is not difficult to see that the function $w_A$ is the symmetrization of $q_A$, i.e.

$$w_A(X)=q_A(X)+q_A(\bar X)-q_A(V).$$

Let us introduce the new functions $\bar q_A(X)=q_A(\bar X)-q_A(V)$, $A\neq\emptyset$. We have $w_A=q_A+\bar q_A$ and

$$\bar q_A(X)=\begin{cases}-1 & \text{if}\quad X\supseteqq A\\ \phantom{-}0 & \text{otherwise.}\end{cases}$$

Let $\mathcal{S}(V)=\mathcal{S}(n)$, $n=|V|$, be the space of all symmetric set functions $f$ such that $f(X)=f(\bar{X})$ for all $X\subseteqq V$ and $f(\emptyset)=0$. Call a set $A\subseteqq V$ even if $A$ is a nonempty set of even cardinality.

**Theorem 4.1.** *The functions $w_A$ for all even $A\subseteqq V$ are linearly independent and consititute a basis of the space $\mathcal{S}(V)$.*

**Proof.** Consider the following alternating sum

$$a_f(S) = \sum_{T\subseteqq S}(-1)^{|T|+1}f(T).$$

Let $a_A(S)$ be $a_f(S)$ for $f=w_A$. Using (2.4) and by direct calculation, it can be shown that

$$\sum_{X\subseteqq B}(-1)^{|X|}\bar{q}_A(X) = (-1)^{|A|+1}\delta(A, B),$$

$$\sum_{X\subseteqq B}(-1)^{X}q_A(X) = \begin{cases}-1 & \text{if} \quad B\subseteqq A, \quad B\neq\emptyset,\\ 0 & \text{otherwise}\end{cases}$$

for $A\neq\emptyset$. Therefore we have

$$a_A(B) = \sum_{X\subseteqq B}(-1)^{|X|+1}w_A(X) = \begin{cases}1 & \text{if} \quad B\subseteqq A, \quad B\neq\emptyset, A\\ 2 & \text{if} \quad B=A \quad \text{and} \quad A \quad \text{is even}\\ 0 & \text{otherwise.}\end{cases} \tag{4.3}$$

The functions $a_A$ for even $A$ are linearly independent, since by (4.3) $a_A(B)=0$ for $|B|\geqq|A|$, $A\neq B$, and $a_A(A)=2$. As the alternating transformation $f\to a_f$ is nondegenerate, the functions $w_A$ are linearly independent, too. There are $2^{n-1}-1$ different functions $w_A$ for even $A$. Since $\dim\mathcal{S}(n)=2^{n-1}-1$ and $w_A\in\mathcal{S}(n)$, we are done. ∎

We now consider only uniform hypergraph $H(V, E_{2k})$ such that $E_{2k}=\{A\subseteqq V\colon |A|=2k\}$ and $k$ is a positive integer. Let $W=\{w_A(B)\colon A\in E_{2k}, B\subseteqq V\}$ be a $\binom{2k}{n}\times 2^n$ matrix whose $A$-row is a set of values of the function $w_A$ on all $B\subseteqq V$. We set $m=2k$ below. Let $W_0$ be a square submatrix of $W$ constituted by columns of $W$ related to $B\in E_m$. Note that $W_0$ is a (0, 1)-matrix whose $(A, B)$-element is $W_0(A, B)=w(t)$, where $t=|A\cap B|$ is integer $0\leqq t\leqq m$, $w(0)=w(m)=0$ and $w(t)=1$ if $1\leqq t\leqq -1$.

**Lemma 4.2.** *The matrix $W_0$ is nonsingular (if $n\neq 2m$) and $W_0^{-1}(A, B)=v(|A\cap B|)$, where $v(t)$ for $t$ integer, $0\leqq t\leqq m$, is determined by the following system of equations*

$$\begin{gathered}\sum_{t=0}^{m} v(t)\binom{i}{t}\binom{n-m+i}{m-t}+v(m-i) = v_0-\delta(i, 0), \quad 0\leqq i\leqq m\\ \sum_{t=0}^{m} v(t)\binom{m}{t}\binom{n-m}{m-t} = v_0.\end{gathered} \tag{4.4}$$

**Proof.** By a symmetry, we can set $W_0^{-1}(A, B)=v(t)$, where $t=|A\cap B|$. We have $1-W_0(A, B)=1$ if $B\subseteqq A$ or $B\subseteqq\bar{A}$ and $1-W_0(A, B)=0$ otherwise. Hence for

$A, B \in E_m$

$$\delta(|A\cap B|, m) = \delta(A, B) = \sum_{C\in E_m} W_0(A, C)W_0^{-1}(C, B) = \sum_{C\in E_m} v(|C\cap B|)-$$

$$-\sum_{C\subseteqq A\vee \bar{A},\, C\in E_m} v(|C\cap B|) = \sum_{C\in E_m} v(|C\cap B|)-v(|A\cap B|)-\sum_{C\subseteqq \bar{A},\, C\in E_m} v(|C\cap B|).$$

Setting $|C\cap B|=t$, $|A\cap B|=m-i$, $v_0=\sum_{C\in E_m} v(|C\cap B|)=\sum_{t=0}^{m}\binom{m}{t}\binom{n-m}{m-t}v(t)$, we obtain (4.4).

We show that the system (4.4) nondegenerate for even $m$. We multiply the $i$-th equality by $(-1)^i\binom{q}{i}$, sum by $i$ over the interval $0\leqq i\leqq q$, and introduce new unknowns $u(t)=\sum_{i=0}^{t}(-1)^l\binom{t}{l}v(l)$. We use two identities

$$\sum_{l=0}^{k}(-1)^l\binom{k}{l}\binom{n+1}{m+k} = (-1)^k\binom{n}{m} \quad\text{and}\quad \sum_{l=0}^{k}\binom{k}{l}\binom{n-k}{m-l} = \binom{n}{m}.$$

This alternating transformation converts the system (4.4) to the following triangular form

$$\sum_{q=t}^{m} a(t, q)u(q) = -1, \quad 0 \leqq t \leqq m,$$

where

$$a(t, q) = (-1)^q\binom{m-t}{q-t}\left[1-\binom{n-q}{m-q}\delta(t, 0)\right]+\binom{n-m-t}{m-t}\delta(t, q), \quad 0 \leqq t \leqq q \leqq m.$$

The diagonal coefficients $a(t, t)$, $0\leqq t\leqq m-1$, are different from zero if $n\neq 2m$. Since $a(m, m)=1+(-1)^m$, the system (4.4) is not degenerate only if $m$ is even and $n\neq 2m$. ∎

Let $D_m(V)$ be the subspace of $\mathscr{S}(V)$ spanned by $w_A$, $A\in E_m$, for even $m=2k$.

**Theorem 4.3.** *The space $D_m(V)$ for even $m$ is determined by the linear equalities*

$$(4.5)\qquad f(B) = \sum_{t=0}^{m} u(b, t)\sum_{A\in E_m,\, |A\cap B|=t} f(A), \quad B\subseteqq V, \quad B\notin E_m,$$

*where $b=|B|$, and*

$$(4.6)\quad u(b, t) = \sum_{q=0}^{m}(-1)^q u(q)\left[\binom{m}{q}\binom{n-q}{m-q}-\binom{t}{q}\binom{b-q}{m-q}-\binom{m-t}{q}\binom{n-b-q}{m-q}\right]$$

*with $u(q)$ from the proof of Lemma* 4.2.

**Proof.** We have $f=\sum_{A\in E_m} c_A w_A$ for $f\in D_m(V)$, or $f=cW$. In particular $f_0=cW_0$, where $f_0$ is a restriction of $f$ onto the family $E_m$. By Lemma 4.2, the matrix $W_0$ is nonsingular for even $m$, so we have $c=f_0W_0^{-1}$. Thus $f=f_0W_0^{-1}W$. We set $u(B, A)=$ $=\sum_{C\in E_m} W_0^{-1}(A, C)W(C, B)$, so that $f(B)=\sum_{A\in E_m} u(B, A)f(A)$. We have $u(B, A)=$

$=\sum_{C\in E_m} v(|A\cap C|)-\sum_{C\subseteq B\vee\bar{B},\, C\in E_m} v(|A\cap C|)$. If we denote $|A\cap C|=l$, $|A\cap B|=t$, $|B|=b$, $|V|=n$, so that $|A\cap\bar{B}|=m-t$, and set $v(l)=\sum_{q=0}^{l}(-1)\binom{l}{q}u(q)$ we can set $u(B,A)=u(b,t)$, where $u(b,t)$ is given by (4.6) with $\binom{a}{b}=0$ for $b>a$. ∎

**Remark 4.4.** *It is easy to verify that* $u(n-b,m-t)=u(b,t)$, *and therefore* $f(B)=f(\bar{B})$.

Let $\mathcal{R}_m(V)=\{f\in\mathcal{S}(V)\colon f=\sum_{A\in E_m} c_A w_A,\, c_A\geqq 0\}$ be the cone spanned by the functions $w_A$, $A\in E_m$.

**Theorem 4.5.** *The cone* $\mathcal{R}_m(V)$ *for even m spans the space* $D_m(V)$. *The facets of the cone* $\mathcal{R}_m(V)$ *are described by the inequalities*

$$\sum_{t=0}^{m} v(t)\sum_{B\in E_m,\,|B\cap A|=t} f(B)\geqq 0 \quad \textit{for all} \quad A\in E_m, \tag{4.7}$$

*which are equivalent (up to the the equalities* (4.5)*) to the following alternating inequalities*

$$\sum_{T\subseteq A}(-1)^{|T|}f(T)\leqq 0 \quad \textit{for all} \quad A\in E_m. \tag{4.8}$$

**Proof.** In the basis $w_A$, the cone $\mathcal{R}_{2k}(V)$ is determined by the inequalities $c_A\geqq 0$ for $A\in E_{2k}$. The proof of Theorem 4.3 implies that $c_A=\sum_{B\in E_{2k}} f(B)W_0^{-1}(B,A)=\sum_{t=0}^{2k} v(t)\sum_{B\in E_{2k},\,|B\cap A|=t} f(B)$. Hence the inequalities $c_A\geqq 0$ are equivalent to the inequalities (4.7).

Alternating the equality $f=\sum c_A w_A$ we obtain $a_f(T)=\sum c_A a_A(T)$. Since by (4.3) $a_A(B)=2\delta(A,B)$ for $A,B\in E_{2k}$, $a_f(A)=2c_A$ for $A\in E_{2k}$. Thus, the inequalities $c_A\geqq 0$ have the form $a_f(A)\geqq 0$ for $A\in E_{2k}$. Using (4.2), we obtain (4.8). ∎

**Remark 4.6.** *For* $m=2k=const$ *Theorems* 4.3 *and* 4.5 *provide a co-NP description of the cone* $\mathcal{R}_m(V)$.

Note that the inequalities (4.8) for $A\in E_2=V^2$ have the form $f(i)+f(j)-f(i,j)\geqq 0$ for $(i,j)\in V^2$. Thus, if a submodular set function lies in the space $D_2(V)$ then it is a cut function. Using an explicite expression for $u(b,t)$ with $m=2$, we obtain

**Corollary 4.7.** *The cone* $\mathcal{R}_2(V)$ *of graphic cut functions is the intersection of the space* $D_2$ *with the cone* $\mathcal{C}(V)$ *of all submodular set functions. The space* $D_2(V)$ *is defined by the equations*

$$f(B)=(n-2)^{-1}(n-4)^{-1}\times$$
$$\times\Big(\bar{b}(\bar{b}-2)\sum_{(i,j)\subseteq B} f(i,j)+b(b-2)\sum_{(i,j)\subseteq\bar{B}} f(i,j)-(b-2)(\bar{b}-2)\sum_{i\in B,\,j\in\bar{B}} f(i,j)\Big)$$

*where* $b=|B|$, $\bar{b}=n-b=|\bar{B}|$. *The facets of* $\mathcal{R}_2(V)$ *are defined by the inequalities*

$$f(i)+f(j)-f(i,j)\geqq 0, \quad (i,j)\in V^2. \;∎$$

## 5. The cone of functions of cuts separating two fixed vertices in a hypergraph

Let $H'(V', E)$ be a hypergraph on the set $V'=V\cup\{s, t\}$, where a source $s$ and a sink $t$ do not belong to $V$. Let $f'=\sum_{A\in E} c_A w_A$ be a cut function of the hypergraph $H'$. We consider a restriction of $f'$ onto sets of the form $\{s\}\cup X$, where $X\subseteq V$. We set $f(X)=f'(s\cup X)$ for all $X\subseteq V$ and call $f$ a function of cuts separating the source $s$ and the sink $t$.

As for $X\subseteq V$, $A\subseteq V'$, $q_A$ of (2.2) and $\bar{q}_A$ of (4.1), it is valid

$$q_A(s\cup X)=\begin{cases}1 & \text{if } A\ni s\\ q_{A-t}(X) & \text{if } A\not\ni s,\end{cases}\qquad \bar{q}_A(s\cup X)=\begin{cases}0 & \text{if } A\ni t\\ \bar{q}_{A-s}(X) & \text{if } A\not\ni t,\end{cases}$$

we have

$$\text{(5.1)}\quad w_A(s\cup X)=q_A(s\cup X)+\bar{q}_A(s\cup X)=\begin{cases}q_{A-t}(X) & \text{if } A\not\ni s,\ A\ni t\\ 1+\bar{q}_{A-s}(X) & \text{if } A\ni s,\ A\not\ni t.\end{cases}$$

Therefore the function $f$ has the form

$$f=\sum_{A\ni s} c_A+\sum_{A\ni s,\, A\not\ni t} c_A\bar{q}_{A-s}+\sum_{A\not\ni s,\, A\ni t} c_A q_{A-t}+\sum_{A\not\ni s,t} c_A w_A.$$

Obviously $f$ belongs to the cone

$$\mathcal{R}(V)=\Big\{f\in\mathbf{R}^{B(V)}\colon f=c_0+\sum_{A\subseteq V,\, A\neq\emptyset}(a_A\bar{q}_A+b_A q_A),\ a_A,\ b_A\geqq 0\Big\}.$$

Note, the cone $\mathcal{R}(V)$ is the subcone of the cone $\mathcal{C}(V)$ of all submodular functions and contains the space $\{f\colon f=c_0+\sum_{i\in V} c_i q_i,\ c_i\in\mathbf{R}\}$ of all modular functions, since $\bar{q}_i=-q_i$. Besides $\mathcal{R}(V)$ contains the alternating cone $\mathcal{A}_0(V)$.

**Theorem 5.1.** *Every $f\in\mathcal{R}(V)$ is up to a constant a function of cuts separating $s$ and $t$ in a hypergraph $H'(V', E)$, where $V'=V\cup\{s, t\}$.*

**Proof.** Let $f=\sum_{A\subseteq V}(a_A\bar{q}_A+b_A q_A)+c_0$. Set $c_A=\min(a_A, b_A)$, $E_1=\{A\subseteq V\colon a_A>c_A\}$, $E_2=\{A\subseteq V\colon b_A>c_A\}$ and $E_3=\{A\subseteq V\colon c_A>0\}$.

According to (5.1), $f$ can be represented in the form

$$f(X)=\sum_{A\in E_1}(a_A-c_A)\big(w_{A\cup s}(s\cup X)-1\big)+$$
$$+\sum_{A\in E_2}(b_A-c_A)w_{A\cup t}(s\cup X)+\sum_{A\in E_3}c_A w_A(s\cup X)+c_0$$

where $X\subseteq V$. This is up to the constant $c_0-\sum_{A\in E_1}(a_A-c_A)$ the function of cuts separating $s$ and $t$ in the hypergraph $H(V', E)$, where $E=E_1'\cup E_2'\cup E_3$ and $E_1'=\{A\cup s\colon A\in E_1\}$, $E_2'=\{A\cup t\colon A\in E_2\}$. ∎

**Theorem 5.2.** *There is a netflow algorithm for the minimization of a submodular set function of the cone $\mathcal{R}(V)$.*

**Proof.** Let $f=c_0+\sum_{A\in E_1}a_A\bar{q}_A+\sum_{B\in E_2}b_B q_B\in\mathcal{R}(V)$. Note, that by (2.2) and (4.1),

$f(X)=c_0-\sum_{A\in E_1, A\subseteq X} a_A+\sum_{B\in E_2, B\cap X\neq\emptyset} b_B$ for $X\subseteq V$. We construct a bipartite network $N_f$ such that finite capacities of cuts separating a source and a sink of the network $N_f$ coincide up to a constant with values of $f$ on all $X\subseteq V$.

Let $E_1$ and $E_2$ be two parts of a vertex set of a bipartite portion of the net $N_f$. The vertices $A\in E_1$ and $B\in E_2$ are adjacent in the graph $N_f$ iff $A\cap B\neq\emptyset$. The capacity of the arc $[A, B]$ is equal to infinity. The source $s$ of the network $N_f$ is adjacent to every vertex $A\in E_1$ by an arc $[s, A]$ of the capacity $a_A$. The sink $t$ is adjacent to every vertex $B\in E_2$ by an arc $[B, t]$ of the capacity $b_B$.

The set of vertices $\{s\}\cup E_1'\cup E_2'$ determines a cut of the network $N_f$ of a finite capacity $\sum_{A\in E_1-E_1'} a_A+\sum_{B\in E_2'} b_B$ iff $A'\cap B'=\emptyset$ for all $A'\in E_1'$ and $B'\in E_2-E_2'$. Hence there is a set $X\subseteq V$ such that $A'\subseteq X$ for all $A'\in E_1'$ and $B'\subseteq\bar{X}$ for all $B'\in E_2-E_2'$. On the other hand, every $X\subseteq V$ determine a finite cut of the above type with $E_1'=\{A\in E_1\colon A\subseteq X\}$ and $E_2'=E_2-\{B\in E_2\colon B\subseteq\bar{X}\}$. The capacity of the cut is

$$\sum_{A\in E_1-E_1'} a_A+\sum_{B\in E_2'} b_B=\sum_{A\in E_1} a_A-\sum_{A\subseteq X} a_A+\sum_{B\colon B\cap X\neq\emptyset} b_B=\sum_{A\in E_1} a_A-c_0+f(X).$$

It follows that a value of a maximum flow in the network $N_f$ is equal to $\min_{X\subseteq V} f(X)$ up to the constant $\sum_{A\in E_1} a_A-c_0$. ∎

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